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Nonlinear stern waves

By J.-M. VANDEN-BROECK

Courant Institute of Mathematical Sciences, New York University, New York, N.Y. 10012†

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Steady two-dimensional potential flow past a semi-infinite flat-bottomed body is considered. This stern flow is assumed to separate tangentially from the body. Gravity waves of finite amplitude occur on the free surface. An exact relation between the amplitude of these waves and the Froude number F is derived. It shows that these waves can exist only for F greater than the value $F^* = 2{\cdot}23$. This is slightly less than the value $F_c = 2{\cdot}26$ at which breaking occurs. For F slightly larger than F^*, the steepness is a multi-valued function of F, indicating the existence of more than one solution for these values of F. In addition, a numerical scheme based on an integro-differential equation formulation is derived to solve the problem in the fully nonlinear case. The shape of the free surface profile is computed for different values of F. As a check on the numerical results, they are shown to satisfy the exact relation between steepness and the Froude number.

1. Introduction

In recent years important progress has been achieved in the computation of two-dimensional nonlinear free-surface flows. For example, Vanden-Broeck & Tuck (1977) and Vanden-Broeck, Schwartz & Tuck (1978) have obtained semi-analytical solutions for the flow of an otherwise uniform stream U past a semi-infinite two-dimensional flat-bottomed body of draft H (see figure 1). Their method is based on an expansion in powers of the Froude number $F = U/(gH)^{\frac{1}{2}}$. It yields stern flows in which the flow rises up along the rear face of the body to a stagnation point at which separation occurs. For small values of the Froude number, these stern flows are physically satisfactory and represent small perturbations from the plane $y = 0$. However, for large values of F, these solutions become physically unreasonable since the ratio of the elevation of the stagnation point to the draft tends to infinity as $F \to \infty$.

In the present paper, we describe analytically and numerically a new family of stern flows in which the flow separates at the corner of the body (see figure 1). These solutions reduce to a uniform stream as $F \to \infty$. As the Froude number decreases from infinity, the steepness s of the waves (i.e. the peak-to-trough wave height divided by the wavelength) increases and reaches the critical 'breaking value' 0·141 for

$$F = F_c = 2{\cdot}26$$

(see figure 2). As F approaches F_c, the steepness of the waves becomes a multi-valued function of the Froude number. From figure 3 we see that the new solution exists only for $F \geqslant F^* = 2{\cdot}23$.

† Present address: Department of Mathematics, Stanford University, Stanford, CA 94305.

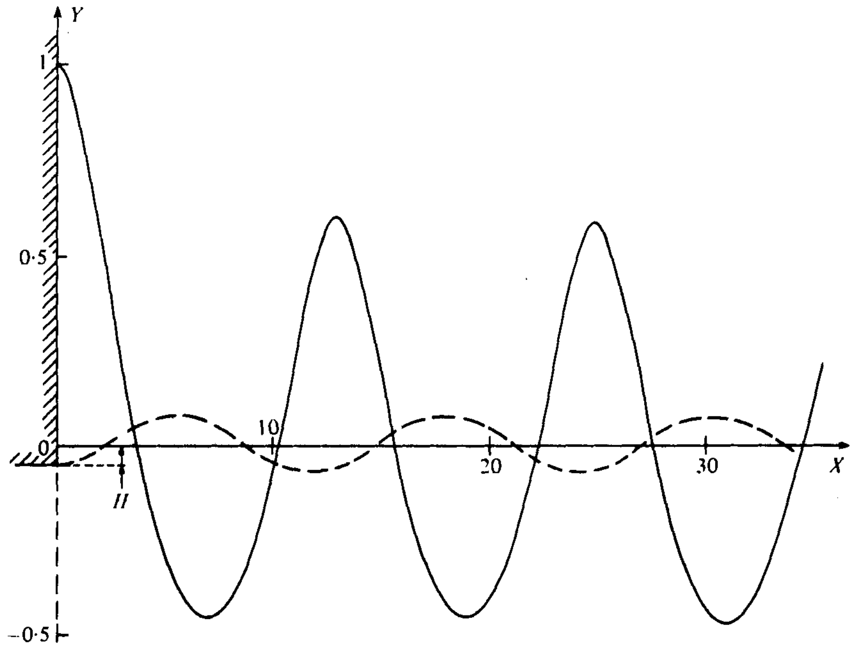

FIGURE 1. Computed free-surface profiles with a stagnation point (solid curve) and with separation (dashed curve) for $F = 6{\cdot}3$. The ordinate $y = 0$ corresponds to the level of the free surface for which the velocity is equal to U. The unit of length is $U^2/2g$. Thus the ordinate of the corner of the body is $-2/F^2$.

Figure 2 shows that the previously obtained solution with a stagnation point is the only solution for $F < F^*$ while there are two or more solutions for $F > F^*$. The stern flow with separation appears to be the only physically acceptable one for F large. Therefore we expect a real stern flow to change from the flow with a stagnation point to the one with separation at some value of the Froude number greater than F^*.

In § 2 we obtain an exact relation between the Froude number and the steepness of the waves. In § 3 we compute nonlinear solutions by a numerical procedure involving an integro-differential equation coupled with Newton's iterations. Typical profiles of the free surface are presented. The numerical results are discussed in § 3.3.

2. Conservation of momentum

The principle of conservation of momentum implies that

$$\int_S \left(\mathbf{V}(\mathbf{V}.\mathbf{n})+gY\mathbf{n}+\frac{p}{\rho}\,\mathbf{n}\right) dS = 0. \tag{2.1}$$

Here S is any closed simply connected contour inside the fluid region, $\mathbf{V}$ is the vector velocity, p is the pressure, ρ is the density and $\mathbf{n}$ is the exterior normal to the contour. We now choose S to consist of the plate S_p, the free surface S_F, a vertical line S_R at $X = +\infty$, a horizontal line S_H at $Y = -\infty$ and a vertical line S_L at $X = -\infty$. We take the component of (2.1) along the X axis. After some algebra we obtain (see appendix)

$$F = U\left(\frac{6g}{\rho}\,V-\frac{4g}{\rho}\,T\right)^{-\frac{1}{2}}. \tag{2.2}$$

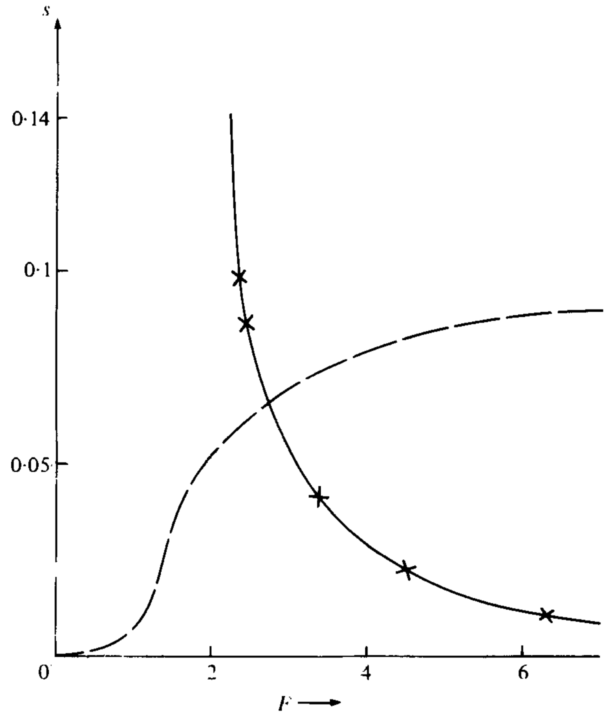

FIGURE 2. Relation between steepness and Froude number. The solid curve shows the theoretical relation (2.2) in which Cokelet's (1977) results are used. The crosses (×) are the numerical results. The dashed line corresponds to Vanden-Broeck & Tuck's (1977) solution.

Here V and T are, respectively, the mean potential energy and the mean kinetic energy per unit horizontal area of the waves far away from the plate. Since the flow is steady, the phase velocity of the waves in the far field is equal to U.

Gravity waves are characterized by their steepness s defined as the difference of ordinates between one crest and one trough divided by the wavelength. The quantities U, V and T are functions of the steepness of the waves. Thus (2.2) is a relation between the Froude number and the steepness of the waves far away from the plate. The mean potential energy, the mean kinetic energy and the phase velocity have been computed as functions of the steepness by Cokelet (1977) using Schwartz's (1974) technique.

Figure 2 shows the relation between steepness and Froude number obtained by substituting Cokelet's results into (2.2). On the same graph, we present the corresponding curve obtained by Vanden-Broeck & Tuck (1977) in the case of the stern flow with a stagnation point. The latter solution appears to exist for all values of $F > 0$. On the other hand, the steepness of the waves in our new solution reaches the critical breaking value 0·141 for $F = F_c = 2{\cdot}26$. In figure 3(*a*), we present the curve corresponding to our new solution on an expanded scale. This figure shows that the steepness of the waves becomes a multi-valued function of the Froude number for F slightly smaller than F_c. It also shows that our solution exists only for $F > F^* = 2{\cdot}23$.

Longuet-Higgins & Fox (1978) have found asymptotic solutions which show that the integral properties of gravity waves oscillate infinitely often as the maximum

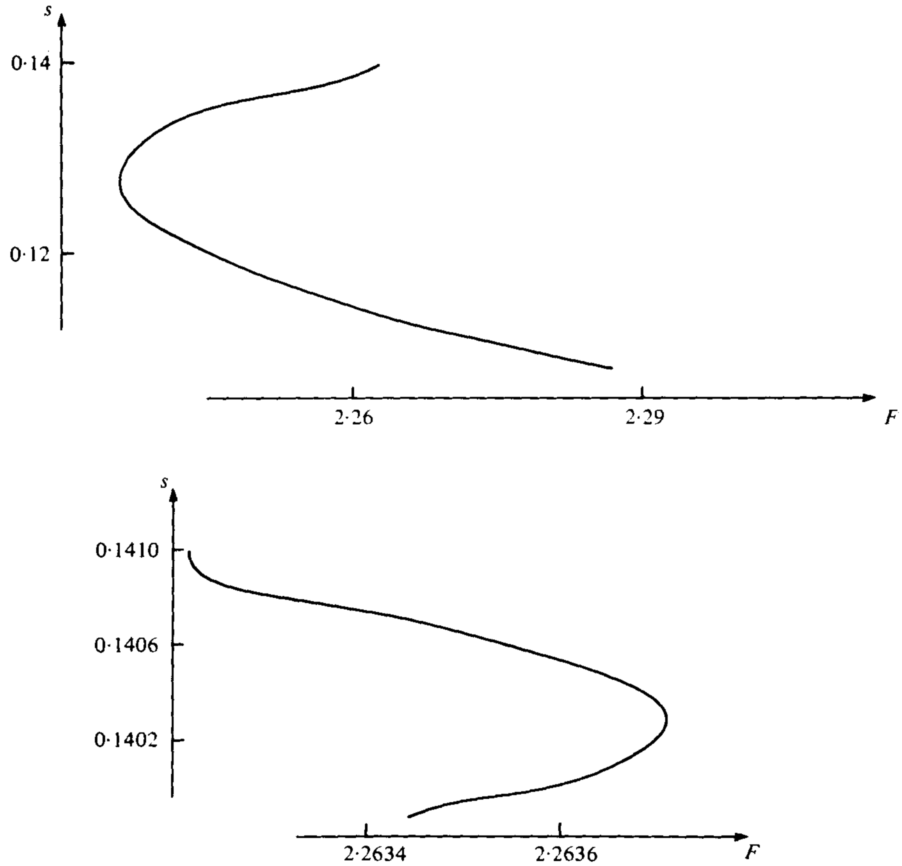

FIGURE 3. (a) The theoretical relation (2.2) between steepness and Froude number in which Cokelet's (1977) results are used. The curve represents a small portion of the corresponding curve in figure 2 on an expanded scale. (b) The theoretical relation between steepness and Froude number in which Longuet-Higgins & Fox's (1978) results are used. The scale has been expanded to show clearly the second oscillation.

height is approached. Substituting their formulas into (2.2), we obtain the parametric equations

$$\begin{aligned} F = {} & [1{\cdot}1931 - 1{\cdot}18\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 2{\cdot}22)]^{\frac{1}{2}} \\ & \times [0{\cdot}05426 - 1{\cdot}014\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 1{\cdot}49) \\ & + 0{\cdot}86\epsilon^3 \cos(2{\cdot}143 \ln \epsilon + 1{\cdot}66)]^{-\frac{1}{2}} + o(\epsilon^3), \end{aligned} \tag{2.3}$$

$$s = 0{\cdot}14107 - \frac{0{\cdot}5}{\pi}\epsilon^2 + 0{\cdot}16\epsilon^3 \cos(2{\cdot}143 \ln \epsilon - 1{\cdot}54) + o(\epsilon^3). \tag{2.4}$$

The parameter ϵ is proportional to the speed at the crest of the wave, which is zero for the highest wave.

Relations (2.3) and (2.4) show that the Froude number oscillates infinitely often as the maximum steepness is approached. The first oscillation is obtained directly from Cokelet's results and is presented in figure 3 (a). The subsequent oscillations are described parametrically by (2.3) and (2.4). The second oscillation is shown in figure

F	s	h/H	λ/H
3·3406	0·04158	2·866	68·93
2·5949	0·07378	2·958	40·09
2·4473	0·08625	3·015	34·97
2·2872	0·10820	3·169	29·29
2·2356	0·12722	3·412	26·82
2·2399	0·13291	3·535	26·59
2·2453	0·13509	3·595	26·61
2·2572	0·13825	3·705	26·80
2·2629	0·13989	3·770	26·95

TABLE 1. Values of s, h/H and λ/H for various values of F

3(*b*). These oscillations indicate the existence of an infinite number of solutions for values of F arbitrarily close to F_c.

Relation (2.2) and the results tabulated by Cokelet were used to obtain the wave height h (i.e. the difference of ordinates between a crest and a trough) and the wavelength λ as functions of the Froude number F. These results are presented in table 1. For large values of F, the quantities s, h/H and λ/H can be evaluated by approximating the waves by the sine waves of linear theory. Thus we obtain

$$s = 2^{\frac{1}{2}}/(\pi F^2), \tag{2.5}$$

$$h/H = 2^{\frac{3}{2}}, \tag{2.6}$$

$$\lambda/H = 2\pi F^2. \tag{2.7}$$

The difference between (2.5), (2.6) and (2.7) and the values of table 1 at $F = 3{\cdot}3406$ is less than 3 %.

3. Numerical solution

3.1. *Formulation*

We denote the potential function by Φ and the stream function by Ψ. Without loss of generality we choose $\Phi = 0$ at the edge of the plate and $\Psi = 0$ on the free surface (and therefore also on the plate). We make the variables dimensionless by referring them to a velocity scale U and a length scale $U^2/2g$.

Thus we define the dimensionless quantities

$$x = \frac{2g}{U^2} X, \tag{3.1}$$

$$y = \frac{2g}{U^2} Y, \tag{3.2}$$

$$\phi = \frac{2g}{U^3} \Phi, \tag{3.3}$$

$$\psi = \frac{2g}{U^3} \Psi. \tag{3.4}$$

The condition of constant pressure on the free surface can be written

$$y+q^2 = 1, \quad \psi = 0, \quad \phi > 0, \tag{3.5}$$

where q is the magnitude of the velocity. In non-dimensional variables we have now a uniform stream with unit velocity at infinity.

If u and v denote respectively the horizontal and the vertical components of the velocity, we write

$$f = \phi + i\psi, \tag{3.6}$$

$$z = x + iy, \tag{3.7}$$

$$u - iv = \left(\frac{dz}{df}\right)^{-1} = \frac{1}{x_\phi + iy_\phi}. \tag{3.8}$$

We shall seek $x_\phi + iy_\phi$ as an analytic function of f, in $\psi \leqslant 0$. The free surface is a portion of the streamline $\psi = 0$, on which (3.5) becomes

$$y + \frac{1}{x_\phi^2 + y_\phi^2} = 1, \quad \psi = 0, \quad \phi > 0. \tag{3.9}$$

We now apply Cauchy's theorem to $x_\phi - 1 + iy_\phi$, on a path consisting of the complete streamline $\psi = 0$ and a semicircle at $\psi = -\infty$. Since $x_\phi - 1 + iy_\phi \to 0$ as $\psi \to -\infty$ we have for $\psi < 0$

$$x_\phi - 1 + iy_\phi = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{(x_\xi - 1 + iy_\xi)|_{\psi=0}}{\xi - f}\, d\xi, \quad \psi < 0. \tag{3.10}$$

Setting $\psi = 0$ in (3.10) and taking the real part, we obtain

$$x_\phi = 1 - \frac{1}{\pi}\int_0^\infty \frac{y_\xi|_{\psi=0}}{\xi - \phi}\, d\xi, \quad \psi = 0, \quad \phi > 0, \tag{3.11}$$

the integral being of Cauchy principal-value form for $\phi > 0$.

Relations (3.9) and (3.11) define a nonlinear singular integro-differential equation for the unknown function $x_\phi + iy_\phi$ on the free surface. We now make the flow leave the plate tangentially by requiring

$$y_\phi|_{\phi=0+,\, \psi=0} = 0. \tag{3.12}$$

The problem is completely defined by imposing the value of y at $\phi = 0$. Using (3.2) and the definition of F we have

$$y(\phi)|_{\phi=0} = -2/F^2. \tag{3.13}$$

3.2. *Numerical analysis*

We seek a numerical solution of the integral equations (3.9), (3.11)–(3.13). The interval of discretization is defined as E. We represent the functions $\partial y/\partial\phi$, $\partial x/\partial\phi$ and $y(\phi)$ on the free surface by the vectors

$$\begin{aligned} \mathbf{y}' &= (y_\phi^1, \ldots, y_\phi^N), \\ \mathbf{x}' &= (x_\phi^1, \ldots, x_\phi^N), \\ \mathbf{y} &= (y^1, \ldots, y^N). \end{aligned} \tag{3.14}$$

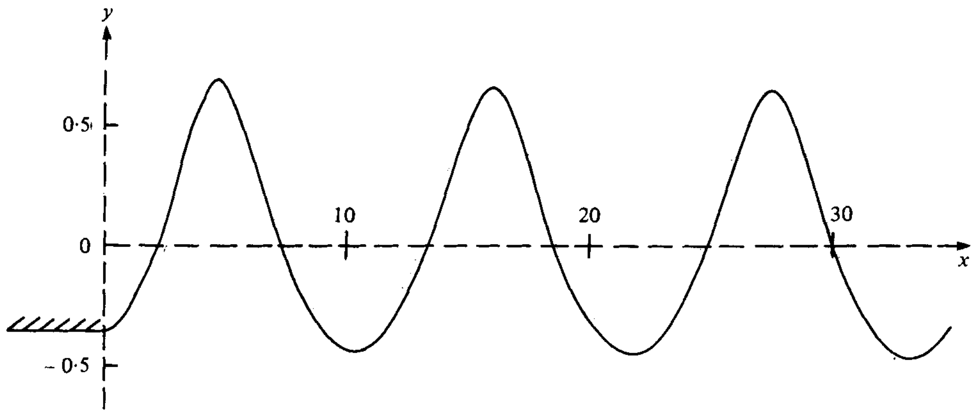

FIGURE 4. Numerical solution for the free surface at $F = 2{\cdot}35$.

The components of these vectors are the values of the functions at the mesh points

$$\phi = \phi_i = (i-1)E, \quad i = 1, \dots, N. \tag{3.15}$$

The condition (3.12) becomes now

$$y^1_\phi = 0.$$

The function $\partial y/\partial \phi$ is approximated between the mesh points by a cubic spline. Using the relations (3.11) and (3.13) we obtain after exact integration of the cubic spline

$$\mathbf{y} = -\frac{2}{F^2} + \mathbf{A}\mathbf{y}', \tag{3.16}$$

$$\mathbf{x}' = 1 + \mathbf{B}\mathbf{y}', \tag{3.17}$$

where $\mathbf{A}$ and $\mathbf{B}$ are known matrices. The error inherent in approximating the infinite integral (3.11) by an integral over a finite interval was found to be negligible at a distance less than a wavelength from the last mesh point.

Substituting (3.16) and (3.17) into (3.9), we get a system of $N-1$ nonlinear algebraic equations with $N-1$ unknowns $(y^2_\phi, \dots, y^N_\phi)$. This system is solved by Newton iterations. For the first approximation we take $\mathbf{y}' = 0$. Each iteration requires the computation and inversion of a matrix. The program was run on a CDC CYBER 173 computer with $N = 100$ and $E = 0{\cdot}5$ and a solution of the algebraic equations with an error less than 10^{-10} was obtained in a few iterations.

3.3. *Discussion of the results*

The free surface profile contains a train of waves behind the plate. The highest point of the profile corresponds to the crest nearest the plate. The steepness of the waves decreases away from the plate and reaches a constant value after a few cycles. Figure 2 shows this constant value for different values of the Froude number F. These numerical results are in good agreement with the theoretical relation derived in § 2.

Typical profiles of the free-surface for $F = 6{\cdot}3$ and $F = 2{\cdot}35$ are presented in figures 1 and 4. In both figures, the horizontal scale has been shrunk in order to show clearly the waves behind the plate. In figure 1 the waves are very close to sine waves. On the other hand in figure 4 the waves are quite noticeably nonlinear with sharp peaks and broad troughs.

The numerical scheme was found to converge only when $F \geqslant 2{\cdot}3$. This domain of convergence coincides with the values of the Froude number for which the steepness s is a single-valued function of F.

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Appendix

Taking the component of (2.1) along the X axis we have

$$\int_S \left[V_X(\mathbf{V}.\mathbf{n}) + gYn_X + \frac{p}{\rho}\,n_X\right] dS = 0. \tag{A 1}$$

Here V_X and n_X are respectively the components of $\mathbf{V}$ and $\mathbf{n}$ along the X axis. It is convenient to replace the line S_H by a horizontal line at $Y = -d$, where d is arbitrarily large. Without loss of generality, we assume that S_R intersects the free surface at the level $Y = 0$.

The integrals over S_p and S_H in (A 1) do not contribute. The integration over S_F, S_R and S_L in (A 1) gives

$$\frac{gH^2}{2} - \int_{-d}^{-H}\left[(V^*)^2 + \frac{p}{\rho} + gY\right]\Bigg|_{X=-\infty} dY - \frac{gd^2}{2} + \frac{S_W}{\rho} = 0. \tag{A 2}$$

Here V^* is the uniform velocity at $X = -\infty$. The quantity S_W is defined by

$$S_W = \int_{-d}^{0} (p + \rho V_x^2)|_{X=+\infty}\, dY \tag{A 3}$$

and represents the momentum flux per unit span of the waves far away from the body. Using Bernoulli's equation we rewrite the integral in (A 2) as

$$\int_{-d}^{-H}\left[(V^*)^2 + \frac{p}{\rho} + gY\right] dY = [\tfrac{1}{2}(U^*)^2 + \tfrac{1}{2}U^2](-H+d). \tag{A 4}$$

For waves in water of infinite depth (A 3) can be written as (Longuet-Higgins 1975)

$$S_W = -3V + \rho dU^2 + \tfrac{1}{2}\rho d^2 g. \tag{A 5}$$

Here V is the mean potential energy per unit horizontal area of the waves. Substituting (A 4) and (A 5) into (A 2) we have

$$\frac{gH^2}{2} = \frac{3V}{\rho} - \tfrac{1}{2}U^2H - \tfrac{1}{2}(V^*)^2H + \tfrac{1}{2}[(V^*)^2d - U^2d]. \tag{A 6}$$

The conservation of mass can be written as

$$V^*(d-H) = Ud - 2T/(U\rho), \tag{A 7}$$

where T is the mean kinetic energy per unit horizontal area of the waves. Multiplying (A 7) by V^* and U and adding the results we have

$$(V^*)^2 d - U^2 d = (V^*)^2 H + V^* UH - 2\frac{T}{\rho} - 2\frac{T}{\rho}\frac{V^*}{U}. \tag{A 8}$$

Substituting (A 8) into (A 6), we obtain

$$\frac{gH^2}{2} = \frac{3V}{\rho} - \frac{T}{\rho} - \frac{T}{\rho}\frac{V^*}{U} + \frac{V^*UH}{2} - \frac{U^2}{2}. \tag{A 9}$$

As $d \to -\infty$, we have $V^* = U$. Thus

$$\frac{gH^2}{2} = \frac{3V}{\rho} - \frac{2T}{\rho}. \tag{A 10}$$

Using the definition of F we rewrite (A 10) as (2.2).

REFERENCES

COKELET, E. D. 1977 Steep gravity waves in water of arbitrary uniform depth. *Phil. Trans. Roy. Soc.* A **286**, 183–230.

LONGUET-HIGGINS, M. S. 1975 Integral properties of periodic gravity waves of finite amplitude. *Proc. Roy. Soc.* A **342**, 157–174.

LONGUET-HIGGINS, M. S. & FOX, M. J. H. 1978 Theory of the almost-highest wave. Part 2. Matching an analytic extension. *J. Fluid Mech.* **85**, 769–786.

SCHWARTZ, L. W. 1974 Computer extension and analytic continuation of Stokes' expansion for gravity waves. *J. Fluid Mech.* **62**, 553–578.

VANDEN-BROECK, J.-M. & TUCK, E. O. 1977 Computation of near-bow or stern flows, using series expansion in the Froude number. In *Proc. 2nd Int. Conf. Num. Ship Hydrodynamics, Berkeley*.

VANDEN-BROECK, J.-M., SCHWARTZ, L. W. & TUCK, E. O. 1978 Divergent low Froude-number series expansion of non-linear free-surface flow problems. *Proc. Roy. Soc.* A **361**, 207–224.